Visualizing elements of order 7 in the Tate-Shafarevich group of an elliptic curve

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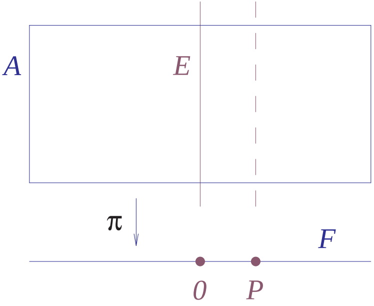
Visibility (Mazur, 1999)

Exact sequence of abelian varieties over $\mathbb{Q}$

$$0 \longrightarrow E \xrightarrow{\iota} A \xrightarrow{\pi} F \longrightarrow 0$$



$$\longrightarrow A(\mathbb{Q}) \longrightarrow F(\mathbb{Q}) \xrightarrow{\delta} H^1(\mathbb{Q}, E) \xrightarrow{\iota_*} H^1(\mathbb{Q}, A) \rightarrow$$

Definition. $\mathrm{Vis}_A H^1(\mathbb{Q}, E) = \mathrm{im}(\delta) = \ker(\iota_*)$

Definition. $Ш(E/\mathbb{Q}) = \ker\left(H^1(\mathbb{Q}, E) \rightarrow \prod_v H^1(\mathbb{Q}_v, E)\right)$

Examples of Visible Ш

We take $A = \dfrac{E \times F}{\Delta}$ $\qquad \left. \begin{array}{l} \Delta \subset E \\ \Delta \subset F \end{array} \right\}$ common finite Galois submodule

Cremona and Mazur (2000)

$\dim E = \dim F = 1 \qquad \Delta = E[n] = F[n]$

$n = 2, 3, 4, 5$

Agashe and Stein (2005)

$\dim E > 1 \quad \dim F = 1 \qquad \Delta = F[n] \subset E[n]$

$n = 3, 5, 7, \ldots, 31$

This talk

$\dim E = 1 \quad \dim F = 1 \qquad \Delta = E[7] = F[7]$

$\dim F = 2 \qquad \Delta = E[7] = F[3 + \sqrt{2}]$

The Visibility Dimension

$E/\mathbb{Q}$ elliptic curve.

Definition. The *visibility dimension* of $\xi \in Ш(E/\mathbb{Q})$ is the least dimension of an abelian variety A such that $\xi \in \mathrm{Vis}_A H^1(\mathbb{Q}, E)$.

- Restriction of scalars shows $\quad \mathrm{vis\,dim}(\xi) \leqslant \mathrm{order}(\xi)$
- Mazur (1999) : $\mathrm{order}(\xi) = 3 \implies \mathrm{vis\,dim}(\xi) \leqslant 2$
- Fisher (2014) : $\exists\, \xi$ of orders 6 and 7 with $\mathrm{vis\,dim}(\xi) > 2$

Observation. The visibility dimension is often much smaller than the bound coming from restriction of scalars.

Examples from Cremona's Tables

$E/\mathbb{Q}$ with $Ш(E/\mathbb{Q})[7] \neq 0$ (and no rational 7-isogeny)

E	F	E	F	E	F
$3364c$	$10092c$	$10800y$	$10800u$	$15219c$	
$6552y$	$6552ba$	$11970o$	$11970s$	$17271g$	
$6622b$		$12927e$	$12927d$	$17816c$	
$7139a$		$13432b$		$18513b$	
$9450p$	$9450t$	$13673a$		$18550c$	
$9510e$	$561090*$	$14938n$		$18832a$	$1712d$

We searched for rational points on twists of the Klein quartic

$$X(7) = \{x^3y + y^3z + z^3x = 0\} \subset \mathbb{P}^2.$$

The appropiate twists are given by formulae of Halberstadt, Kraus and Poonen, Schaefer, Stoll.

Examples from Cremona's Tables

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E	F	E	F	E	F
$3364c$	$10092c$	$10800y$	$10800u$	$15219c$	
$6552y$	$6552ba$	$11970o$	$11970s$	$17271g$	x
$6622b$		$12927e$	$12927d$	$17816c$	x
$7139a$	x	$13432b$	x	$18513b$	x
$9450p$	$9450t$	$13673a$	x	$18550c$	x
$9510e$	$561090\,*$	$14938n$		$18832a$	$1712d$

In the cases indicated we found an elliptic curve $F/\mathbb{Q}$ with $E[7] \cong F[7]$ and rank $F(\mathbb{Q}) = 2$. Therefore

$$(\mathbb{Z}/7\mathbb{Z})^2 \cong \frac{F(\mathbb{Q})}{7F(\mathbb{Q})} \hookrightarrow H^1(\mathbb{Q}, E).$$

First Example

$E = 6622b \qquad N_E = 6622 = 2 \times 7 \times 11 \times 43.$

There are 10 isogeny classes of this conductor, but the elliptic curves in the other 9 isogeny classes are not 7-congruent to E.

However we find $f \in S_2(\Gamma_0(6622))$ with

p	2	3	5	7	11	13
$a_p(E)$	-1	2	2	-1	1	6
$a_p(f)$	-1	$-\sqrt{2}-1$	$\sqrt{2}-2$	-1	1	$2\sqrt{2}-2$

$$a_p(f) \equiv a_p(E) \mod (3+\sqrt{2}) \qquad \text{for all } p$$

Question. Can we find $C/\mathbb{Q}$ genus 2 curve with

$$\mathrm{Trace}(a_p(f)) = p + 1 - n_1$$
$$\mathrm{Norm}(a_p(f)) = (n_1^2 + n_2)/2 - (p+1)n_1 - p$$

where $n_i = \#\widetilde{C}(\mathbb{F}_{p^i})$?

First Example (continued)

$E = 6622b \qquad N_E = 6622 = 2 \times 7 \times 11 \times 43.$

Question. Can we find $C/\mathbb{Q}$ genus 2 curve with

$$\begin{aligned} \mathrm{Trace}(a_p(f)) &= p + 1 - n_1 \\ \mathrm{Norm}(a_p(f)) &= (n_1^2 + n_2)/2 - (p+1)n_1 - p \end{aligned}$$

where $n_i = \#\widetilde{C}(\mathbb{F}_{p^i})$?

Answer. Yes.

$$\begin{aligned} y^2 &= 20x^6 + 44x^5 - 23x^4 - 10x^3 + 81x^2 - 52x + 4 \\ &= \mathrm{Norm}_{K/\mathbb{Q}}\left((-\alpha + 1)x^2 - \frac{\alpha^2 + \alpha}{2} x + \alpha + 3 \right) \end{aligned}$$

where $K = \mathbb{Q}(\alpha)$ and $\alpha^3 + \alpha^2 + \alpha + 17 = 0$.

Using formulae of Bending (1998) we found 35 similar examples with $N_E < 10^5$ and F a genus 2 Jacobian with real multiplication by $\sqrt{2}$.

Remarks.

- These examples were found *without* having to compute any modular forms. However in most cases $N_F = N_E^2$ (up to powers of 2).
- In all but 3 cases we found rank $F(\mathbb{Q}) = 4$. Therefore

$$(\mathbb{Z}/7\mathbb{Z})^2 \cong \frac{F(\mathbb{Q})}{(3+\sqrt{2})F(\mathbb{Q})} \hookrightarrow H^1(\mathbb{Q}, E).$$

Favourite Example

$E = 67080r \qquad N_E = 67080 = 2^3 \times 3 \times 5 \times 13 \times 43.$
We find $f \in S_2(\Gamma_0(13416))$ with

p	2	3	5	7	11	13
$a_p(E)$	0	-1	-1	2	4	-1
$a_p(f)$	0	-1	$-\sqrt{2}-2$	$\sqrt{2}-2$	$-2\sqrt{2}-2$	-1

$$a_p(E) \equiv a_p(f) \mod (3+\sqrt{2}) \qquad \text{for all } p \neq 5$$

Genus 2 curve $C/\mathbb{Q}$: $\quad y^2 = x(x+4)(x^4+2x^3-x-3).$

Following Poonen, Schaefer, Stoll:
$J = \mathrm{Jac}(X_E(7))$: $\quad \mathrm{rank}\, J(\mathbb{Q}) = 2 < 3 = \dim J \quad$ (under GRH)
Chabauty + Mordell-Weil sieve gives $\# X_E(7)(\mathbb{Q}) = 2$.

Conclusion. Every non-trivial element of $Ш(E/\mathbb{Q}) \cong (\mathbb{Z}/7\mathbb{Z})^2$ has visibility dimension exactly 3.

Checking the Congruences

We must prove $E[7] = F[3+\sqrt{2}]$ as Galois modules.

Possible methods.

1. Fix F. Find the twist of the Klein quartic parametrising the elliptic curves E with $E[7] = F[3+\sqrt{2}]$.
2. Use modularity (Ribet, Khare, Wintenberger)
   Bottlenecks : Computing modular forms.
   Computing 2-part of conductor.
   Possible variant : Use Faltings-Serre method.
3. Exhibit torsion points on $E/\pm 1$ and $F/\pm 1$ with the same field of definition. (Following Kraus, Oesterle (1992)).

We used method 3.

Checking Local Solubility

Using $F/\mathbb{Q}$ with dim $F = 1$ or 2 we have exhibited

$$(\mathbb{Z}/7\mathbb{Z})^2 \hookrightarrow H^1(\mathbb{Q}, E).$$

Question. Do we get elements of $\text{Ш}(E/\mathbb{Q})$?

We have $\Delta = E[7] = F[\phi]$ where $\phi = 7$ or $3 + \sqrt{2}$.

The Selmer groups $S^{(7)}(E/\mathbb{Q})$ and $S^{(\phi)}(F/\mathbb{Q})$ are subgroups of $H^1(\mathbb{Q}, \Delta)$ defined by local conditions.

If these local conditions match up, we do get elements of $\text{Ш}(E/\mathbb{Q})$. This is true in all cases checked so far.

Examples from Cremona's Tables (revisited)

$E/\mathbb{Q}$ with $Ш(E/\mathbb{Q})[7] \neq 0$ (and no rational 7-isogeny)

E	F	E	F	E	F
$3364c$	$10092c$	$10800y$	$10800u$	$15219c$	$\boxed{\text{genus 2}}$
$6552y$	$6552ba$	$11970o$	$11970s$	$17271g$	x
$6622b$	$\boxed{\text{genus 2}}$	$12927e$	$12927d$	$17816c$	x
$7139a$	x	$13432b$	x	$18513b$	x
$9450p$	$9450t$	$13673a$	x	$18550c$	x
$9510e$	$561090 *$	$14938n$	$\boxed{\text{genus 2}}$	$18832a$	$1712d$

The examples with $F/\mathbb{Q}$ a genus 2 Jacobian have

$$E[7] \cong F[3 + \sqrt{2}].$$

Further Examples (in progress)

$E/\mathbb{Q}$ with $Ш(E/\mathbb{Q})[11] \neq 0$

E	F	E	F	E	F
8350c	genus 2	36166j		52416dm	52416dl
13790a	genus 2	36762h	x	55660v	
14570c		38088t	38088u	56144v	genus 2
20806a	x	40755n	x	58029f	genus 2
30940a	x	44082c	genus 2	58558e	genus 2
36076b	x	49450k	x	61275j	x

The examples with $F/\mathbb{Q}$ a genus 2 Jacobian have

$$E[11] \cong F[4-\varphi] \qquad \text{where } \varphi = \frac{1+\sqrt{5}}{2}.$$